The Picard group, the figure-eight knot group and Jorgensen groups (Hyperbolic Spaces and Discrete Groups)

数理解析研究所講究録 1223 巻 2001 年 37-42

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Hiroki Sato

佐藤 宏樹*

Department of Mathematics, Faculty of Science

Shizuoka University

0. Introduction.

In this paper we will state that the Picard group G_P and the figure-eight knot group G_F are two-generator groups and Jørgensen groups. Furthermore we will describe a complete set of relations for G_P as a two-generator group. The detail will appear elsewhere.

1. The Picard group.

DEFINITION 1.1. The group

$$G_P := \left\{ \frac{az+b}{cz+d} \;\middle|\; a, b, c, d \in \mathbf{Z} + i\mathbf{Z}, ad - bc = 1 \right\}$$

is the *Picard group*.

*Partly supported by the Grants-in-Aid for Scientific and Co-operative Research, the Ministry of Education, Science, Sports, Culture and Technology, Japan

2000 *Mathematics Subject Classification.* Primary 32G15; Secondary 20H10, 30F40.

Theorem A (Magnus [7]) *The Picard group G_P is generated by the following four Möbius transformations S_m, T_m, U_m and V_m with corresponding matrices*

$$S_m = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad T_m = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad U_m = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad V_m = \begin{pmatrix} i & -1 \\ 0 & -i \end{pmatrix}.$$

Theorem B (Johnson-Weiss [3]) *The Picard group G_P is generated by the following three matrices*:

$$B_j = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad C_j = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}, \quad S_j = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

See Johnson-Kellerhals-Ratcliffe-Tschantz [1] and Johnson-Weiss [2] for more informations about the Picard group and Coxeter groups.

2. Jørgensen groups.

Theorem C (Jørgensen [4]). *If $\langle A, B\rangle$ is a non-elementary discrete subgroup of* Möb, *then*

$$J(A,B) := |\mathrm{tr}^2(A) - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2| \geq 1.$$

The lower bound 1 *is best possible.*

Definition 2.1. Let A and B be Möbius transformations. The *Jørgensen number* $J(A,B)$ is

$$J(A,B) := |\mathrm{tr}^2(A) - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2|.$$

Definition 2.2. A non-elementary two-generator discrete subgroup G of Möb is a *Jørgensen group* if G has generators A and B with $J(A,B) = 1$.

Theorem D (Jørgensen-Kiikka [5]). *Let $\langle A, B\rangle$ be a non-elementary discrete group with $J(A,B) = 1$, that is, a Jørgensen group. Then A is elliptic of order at least seven or A is parabolic.*

Here we only consider the case where A is parabolic, that is, Jørgensen groups of parabolic type. Namely, we consider two-generator groups $G_{ik,\sigma} = \langle A, B_{ik,\sigma}\rangle$ generated by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{ik,\sigma} = \begin{pmatrix} ik\sigma & -k^2\sigma - 1/\sigma \\ \sigma & ik\sigma \end{pmatrix},$$

where $k \in \mathbf{R}$ and $\sigma \in \mathbf{C} \setminus \{0\}$.

Let C be the following cylinder: $C = \{(\sigma, ik) \mid |\sigma| = 1,\ k \in \mathbf{R}\}$.

THEOREM E (Sato [**9**]). *Every Jørgensen group of type $G_{ik,\sigma}$ lies on the cylinder C.*

By Theorem E we consider two-generator groups $G_{\mu,\sigma} = \langle A, B_{\mu,\sigma}\rangle$ with $\mu = ik$ $(k \in \mathbf{R})$ and $\sigma = -ie^{i\theta}$ $(0 \le \theta < 2\pi)$. For simplicity we set $B_{k,\theta} := B_{ik,\sigma}$ and $G_{k,\theta} = \langle A, B_{k,\sigma}\rangle$ for $\sigma = -ie^{i\theta}$.

We can see that it suffices to consider the case of $(0 \le \theta \le \pi/2)$ and $k \ge 0$.

THEOREM F (Jørgensen-Lascurain-Pignataro [**6**], Sato [**9**], Sato-Yamada [**12**]). *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{k,\theta} = \begin{pmatrix} ke^{i\theta} & ie^{-i\theta}(k^2e^{2i\theta} - 1) \\ -ie^{i\theta} & ke^{i\theta} \end{pmatrix}$$

and let $G_{k,\theta} = \langle A, B_{k,\theta}\rangle$ be the group generated by A and $B_{k,\theta}$, where $k \in \mathbf{R}$ and $\sigma \in \mathbf{C} \setminus \{0\}$. *Then*

(i) *$G_{1/2,\pi/2}$ is a Jørgensen group.*

(ii) *$G_{\sqrt{3}/2,\pi/6}$ is a Jørgensen group.*

See Sato [**9,10**] for Jørgensen groups of parabolic type.

3. Theorems.

In this section we will state main theorems. We can prove the theorems by using Poincaré's polyhedron theorem (cf. Maskit [**8**]).

THEOREM 1 (Sato [**9**,**11**]) (i) *The Picard group G_P is conjugate to $G_{1/2,\pi/2}$, that is, $G_P = RG_{1/2,\pi/2}R^{-1}$, where*

$$R = \begin{pmatrix} 1 & i/2 \\ 0 & 1 \end{pmatrix}$$

(ii) *The following relations form a complete set of relations for G_P :*

$$(B^{-1}ABA^2BAB^{-1}A^2B^{-1}ABA^2BAB^{-1}AB)^2 = 1$$

$$(B^{-1}ABA^2BAB^{-1}A^2B^{-1}ABA)^2 = 1$$

$$(AB^{-1}ABA^2BAB^{-1}A^2B^{-1}ABA^2BAB^{-1}AB)^2 = 1$$

$$(AB^{-1}ABA^2BAB^{-1}A^2B^{-1}ABA)^2 = 1$$

$$(B^{-1}ABA)^3 = 1$$

$$(AB^{-1}ABA)^2 = 1$$

$$(AB^{-1}ABA^2B^{-1}ABA^2BAB^{-1}A^2B^{-1}ABA^2BAB^{-1}AB)^2 = 1$$

$$(AB^{-1}ABA^2B^{-1}ABA^2BAB^{-1}A^2B^{-1}ABA)^3 = 1,$$

where $B = RB_{1/2,\pi/2}R^{-1}$.

COROLLARY. *The Picard group is a two-generator group and a Jørgensen group.*

THEOREM 2 (Sato [**9**,**11**]). *The figure-eight knot group G_F is conjugate to $G_{\sqrt{3}/2,\pi/6}$, that is, $G_F = RG_{\sqrt{3}/2,\pi/6}R^{-1}$, where*

$$R = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$$

(ii) *The following relation forms a complete set of relations for* G_F :

$$ABA^{-1}B^{-1}A = BA^{-1}B^{-1}ABA,$$

where $B = RB_{\sqrt{3}/2,\pi/6}R^{-1}$.

COROLLARY. *The figure-eight knot group is a two-generator group and a Jørgensen group.*

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Department of Mathematics
Faculty of Science
Shizuoka University
Ohya Shizuoka 422-8529
Japan
e-mail:smhsato@ipc.shizuoka.ac.jp